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Approximation of Elements of a Generalized Orlicz Sequence Space by Nonlinear, Singular Kernels

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Let l^φ be a generalized Orlicz sequence space generated by a modular $\rho(x) = \sum_{i=0}^{\infty} \varphi_i(|t_i|)$, $x = (t_i)$, with s-convex functions φ_i, $0 < s \leqslant 1$, and let $K_{w,j}: R_+ \to R_+$ for $j = 0, 1, 2,\ldots$, $w \in \mathscr{W}$, where $\mathscr{W}$ is an abstract set of indices. Assuming certain singularity assumptions on the nonlinear kernel $K_{w,j}$ and setting $T_w x = ((T_w x)_i)_{i=0}^{\infty}$, with $(T_w x)_i = \sum_{j=0}^{i} K_{w,i-j}(|t_j|)$ for $x = (t_j)$, convergence results: $T_w x \to x$ in l^φ are obtained (both modular convergence and norm convergence), with respect to a filter $\mathfrak{M}$ of subsets of the set $\mathscr{W}$.

In [1] a general approximation theorem in modular spaces was obtained and applied to translation operators and linear integral operators in generalized Orlicz spaces L^φ of periodic functions as well as in generalized Orlicz spaces l^φ of sequences (see also [3, Sects. 5 and 7]). The application in L^φ was extended in [2] to some nonlinear integral operators. The aim of this note is to obtain an extension of the result of [1] to the case of approximation by some nonlinear operators in the generalized Orlicz space l^φ of sequences.

Let ρ be an s-convex modular in a real vector space X, $0 < s \leqslant 1$, i.e., $0 \leqslant \rho(x) \leqslant \infty$, $\rho(x) > 0$ for $x \neq 0$, $\rho(0) = 0$, $\rho(-x) = \rho(x)$, and $\rho(ax + by) \leqslant a^s \rho(x) + b^s \rho(y)$ for $a, b \geqslant 0$, $a^s + b^s = 1$. Let $X_\rho = \{x \in X: \rho(ax) \to 0$ as $a \to 0+\}$ be the modular space generated by ρ. Then $\|x\|_\rho = \inf\{u > 0: \rho(x/u^{1/s}) \leqslant 1\}$ is an s-homogeneous F-norm in X_ρ, and the convergence $\|x_n - x\|_\rho \to 0$ as $n \to \infty$ is equivalent to the condition $\rho(a(x_n - x)) \to 0$ as $n \to \infty$ for every $a > 0$ and is denoted by $x_n \to x$. Besides this convergence, a modular convergence or ρ-convergence $x_n \to^\rho x$ is defined in X_ρ by the condition $\rho(a(x_n - x)) \to 0$ as $n \to \infty$ for some $a > 0$ depending on (x_n). If $\mathscr{W}$ is an abstract set of indices and $\mathfrak{M}$ is a filter of subsets of $\mathscr{W}$, then norm

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convergence $x_w \to^{\mathfrak{M}} x$ and ρ-convergence $x_w \to^{\rho,\mathfrak{M}} x$ with respect to the filter $\mathfrak{M}$ is defined analogously (for these definitions, see, e.g., [3, Sects. 1 and 5]). A special case of a modular space is obtained, taking as X the space of all real sequences and defining ρ by the formula

$$\rho(x) = \sum_{i=1}^{\infty} \varphi_i(|t_i|),$$

where $(\varphi_i)_{i=0}^{\infty}$ is a sequence of nonnegative, s-convex functions, i.e., $\varphi_i(au+bv) \leqslant a^s\varphi_i(u) + b^s\varphi_i(v)$, $a, b \geqslant 0$, $a^s + b^s = 1$, with $\varphi_i(0) = 0$, $\varphi_i(u) > 0$ for $u > 0$, $\varphi_i(u) \uparrow \infty$ as $u \uparrow \infty$. The respective modular space $l^\varphi = X_\rho$ is called the generalized Orlicz sequence space (see, e.g., [3, 7.2]). The sequence $(\varphi_i)_{i=0}^{\infty}$ is called τ_+-bounded, if there exist constants $k_1, k_2 \geqslant 1$ and a double-sequence $(\varepsilon_{i,j})$ such that $\varphi_{i+j}(u) \leqslant k_1\varphi_i(k_2 u) + \varepsilon_{i,j}$ for $u \geqslant 0$, $i, j = 0, 1, 2, \ldots$, where $\varepsilon_{i,j} \geqslant 0$, $\varepsilon_{i,0} = 0$, $\varepsilon_j = \sum_{j=0}^{\infty} \varepsilon_{i,j} \to 0$ as $j \to \infty$, $\varepsilon = \sup_{j \geqslant 0} \varepsilon_j < \infty$ (see [3, 7.18]).

Let $\mathfrak{M}$ be a filter of subsets of an abstract set $\mathscr{W}$ of indices and let $T_w\colon X_\rho \to X$ be a family of operators. The family $T = (T_w)_{w \in \mathscr{W}}$ is called $\mathfrak{M}$-bounded, if there are constants $k_1, k_2 > 0$ and a map $g\colon \mathscr{W} \to R_+ = [0, \infty)$ such that $g(w) \to^{\mathfrak{M}} 0$ and for every $x, y \in X_\rho$ there exists a set W depending on $c(x-y)$ such that $\rho(c(T_w x - T_w y)) \leqslant k_1 \rho(ck_2(x-y)) + g(w)$ for every $w \in W$ and $c > 0$ (see [2, p. 456]). The following theorem was proved in [2]:

THEOREM (T). *Let $T = (T_w)_{w \in \mathscr{W}}$ be $\mathfrak{M}$-bounded and let S be a subset of elements $x \in X_\rho$ whose elements satisfy the condition $T_w x \to^{\mathfrak{M}} x$ in X_ρ. If S is ρ-dense in X_ρ, then $T_w x \to^{\rho,\mathfrak{M}} x$ for every $x \in X_\rho$. If S is dense in X_ρ with respect to the s-homogeneous F-norm $\| \ \|_\rho$, then $T_w x \to^{\mathfrak{M}} x$ for every $x \in X_\rho$.*

Now, let for every $w \in \mathscr{W}$, $K_{w,j}\colon R_+ \to R_+$ for $j = 0, 1, 2, \ldots$, and $K_{w,j}(0) = 0$ for all w and j. For any sequence $x = (t_j)_{j=0}^{\infty}$ we define

$$(T_w x)_i = \sum_{j=0}^{i} K_{w,i-j}(|t_j|) \qquad \text{and} \qquad T_w x = ((T_w x)_i)_{i=0}^{\infty}. \tag{1}$$

We shall be interested in obtaining conditions on $K = (K_{w,j})_{j=0}^{\infty}$, $w \in \mathscr{W}$, under which $T_w x \to^{\rho,\mathfrak{M}} x$ or $T_w x \to^{\mathfrak{M}} x$ in the generalized Orlicz sequence space l^φ.

We shall call K a semiregular kernel, if the following conditions are satisfied, where

$$L_{w,i} = \sup_{u \neq v} \frac{|K_{w,i}(u) - K_{w,i}(v)|}{|u - v|}:$$

(1°) $L(w) = (\sum_{i=0}^{\infty} L_{w,i}^s)^{1/s} \leqslant \sigma < \infty$,

(2°) $L_{w,j}/L(w) \to^{\mathfrak{M}} 0$ for $j = 1, 2, \ldots$.

If, moreover, $(1/c)\,K_{w,0}(c) \to^{\mathfrak{M}} 1$ for every $c>0$, then K will be called a singular kernel.

Let us remark that if $K_{w,j}$ are linear, i.e., $K_{w,j}(t)=K_{w,j}t$, and singular in the sense of [1, p. 109], then K is also a singular kernel in our sense.

THEOREM 1. *If K is a semisingular kernel and the sequence $\varphi=(\varphi_i)_{i=0}^{\infty}$ of s-convex functions, $0<s\leqslant 1$, is τ_+-bounded, then the family of operators $T=(T_w)_{w\in\mathscr{W}}$ defined by* (1) *is $\mathfrak{M}$-bounded in the generalized Orlicz sequence space l^{φ} and $T_w\colon l^{\varphi}\to l^{\varphi}$ for every $w\in\mathscr{W}$.*

Proof. Let $c>0$ be arbitrary. Then for $x=(t_j)_{j=0}^{\infty}$, $y=(s_j)_{j=0}^{\infty}$, we have

$$\rho(c(T_w x - T_w y)) \leqslant \sum_{i=0}^{\infty} \varphi_i\left\{c\sum_{j=0}^{i} |K_{w,i-j}(t_j)-K_{w,i-j}(s_j)|\right\}$$

$$\leqslant \sum_{i=0}^{\infty} \varphi_i\left(c\sum_{j=0}^{i} L_{w,j}\,|t_{i-j}-s_{i-j}|\right).$$

But, by s-convexity of φ_i, we obtain

$$\varphi_i\left(c\sum_{j=0}^{i} L_{w,j}\,|t_{i-j}-s_{i-j}|\right)$$

$$\leqslant \frac{1}{L^s(w)}\sum_{j=0}^{i} L_{w,j}^s\varphi_i\left\{\left(\sum_{k=0}^{i} L_{w,k}^s\right)^{-1/s}\right.$$

$$\left.\times\sum_{l=0}^{i} L_{w,l}cL(w)\,|t_{i-1}-s_{i-1}|\right\}$$

$$\leqslant \frac{1}{L^s(w)}\sum_{j=0}^{i} L_{w,j}^s\varphi_i(cL(w)\,|t_{i-j}-s_{i-j}|).$$

Hence

$$\rho(c(T_w x - T_w y)) \leqslant \frac{1}{L^s(w)}\sum_{j=0}^{\infty}\sum_{i=j}^{\infty} L_{w,j}^s\varphi_i(cL(w)\,|t_{i-j}-s_{i-j}|)$$

$$=\frac{1}{L^s(w)}\sum_{j=0}^{\infty} L_{w,j}^s\sum_{i=0}^{\infty}\varphi_{i+j}(cL(w)\,|t_i-s_i|).$$

However, by τ_+-boundedness of $\varphi=(\varphi_i)_{i=0}^{\infty}$, we have

$$\varphi_{i+j}(cL(w)\,|t_i-s_i|)\leqslant k_1\varphi_i(k_2cL(w)\,|t_i-s_i|)+\varepsilon_{i,j}.$$

Thus,

$$\rho(c(T_w x - T_w y)) \leqslant \frac{k_1}{L^s(w)} \sum_{j=0}^{\infty} L^s_{w,j} \sum_{i=0}^{\infty} \varphi_i(k_2 c L(w) \,|t_i - s_i|)$$
$$+ \frac{1}{L^s(w)} \sum_{j=0}^{\infty} L^s_{w,j} \sum_{i=0}^{\infty} \varepsilon_{i,j}$$
$$\leqslant k_1 \rho(k_2 c\sigma(x-y)) + g(w),$$

where

$$g(w) = \frac{1}{L^s(w)} \sum_{j=0}^{\infty} L^s_{w,j}\varepsilon_j = \frac{1}{L^s(w)} \sum_{j=1}^{\infty} L^s_{w,j}\varepsilon_j,$$

because $L_{w,0} = 0$, by the assumption $\varepsilon_{i,0} = 0$.

In order to prove the theorem it is sufficient to show that $g(w) < \infty$ for all $w \in \mathscr{W}$ and $g(w) \to^{\mathfrak{M}} 0$.

Let us fix an $\eta > 0$. Then there exists an index r such that $\sup_{j>r} \varepsilon_j < \frac{1}{2}\eta$. Then

$$0 \leqslant g(w) = \frac{1}{L^s(w)} \sum_{j=1}^{r} L^s_{w,j}\varepsilon_j + \frac{1}{L^s(w)} \sum_{j=r+1}^{\infty} L^s_{w,j}\varepsilon_j$$
$$\leqslant \frac{1}{L^s(w)} \sum_{j=1}^{r} L^s_{w,j}\varepsilon + \frac{1}{2}\eta < \infty \qquad \text{for all } w \in \mathscr{W}.$$

Now, let us choose $W \in \mathfrak{M}$ so that

$$\frac{L_{w,j}}{L(w)} < \left(\frac{\eta}{2\varepsilon r}\right)^{1/s} \qquad \text{for} \quad j = 1, 2, \ldots, r.$$

Then there holds for $w \in W$

$$0 \leqslant g(w) \leqslant \varepsilon \sum_{j=1}^{r} \frac{\eta}{2\varepsilon r} + \frac{1}{2}\eta = \eta,$$

whence $g(w) \to^{\mathfrak{M}} 0$. Thus, we conclude the theorem.

Now, given a kernel K defined by (1) and a number $c > 0$, let us denote

$$x^j_w(c) = (\underbrace{0, 0, \ldots, 0}_{j+1 \text{ times}}, K_{w,1}(c), K_{w,2}(c), \ldots). \tag{2}$$

Moreover, let us write $e_k = (\delta_{i,k})_{i=0}^{\infty}$ with $\delta_{i,k} = 1$ for $i = k$, $\delta_{i,k} = 0$ for $i \neq k$. Then there holds the following

LEMMA. *If* $x = c_0 e_0 + c_1 e_1 + \cdots + c_n e_n$ *and* φ_i *are s-convex,* $0 < s \leqslant 1$, *then for every* $b > 0$ *there holds the inequality*

$$\rho(b2^{-1/s}(n+1)^{-1/s}(T_w x - x)) \leqslant \tfrac{3}{2} \sum_{j=0}^{n} \rho(bx_w^j(c_j)) + \tfrac{1}{2} \sum_{j=0}^{n} \varphi_j(b\,|K_{w,0}(c_j) - c_j|). \tag{3}$$

Proof. It is easily seen that

$$(T_w x - x)_i = \sum_{j=0}^{i} K_{w,i-j}(c_j) - c_i \qquad \text{for} \quad i \leqslant n,$$

$$= \sum_{j=0}^{n} K_{w,i-j}(c_j) \qquad \text{for} \quad i > n.$$

Hence for any $a > 0$,

$$\rho(a(T_w x - x)) = \sum_{i=0}^{n} \varphi_i \left\{ a \left| \sum_{j=0}^{i} K_{w,i-j}(c_j) - c_i \right| \right\} + \sum_{i=n+1}^{\infty} \varphi_i \left\{ a \sum_{j=0}^{n} K_{w,i-j}(c_j) \right\}. \tag{4}$$

From s-convexity of φ_i we have

$$\varphi_i \left(a \sum_{j=0}^{k} a_{i,j} \right) \leqslant \frac{1}{k+1} \sum_{j=0}^{k} \varphi_i(a(k+1)^{1/s} a_{i,j}) \tag{5}$$

for $a_{i,j} \geqslant 0$ and arbitrary $k = 0, 1, 2, \ldots$.

Taking $k = i$, $a_{i,j} = K_{w,i-j}(c_j)$ for $0 \leqslant j < i$, $a_{i,i} = K_{w,0}(c_i) - c_i$ in (5), we obtain after easy calculations,

$$\sum_{i=0}^{n} \varphi_i \left\{ a \left| \sum_{j=0}^{i} K_{w,i-j}(c_j) - c_i \right| \right\} \leqslant \tfrac{1}{2} \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} \varphi_i \{ 2^{1/s} a(n+1)^{1/s} K_{w,i-j}(c_j) \} + \tfrac{1}{2} \sum_{i=0}^{n} \varphi_i \{ 2^{1/s} a(n+1)^{1/s} |K_{w,0}(c_i) - c_i| \}, \tag{6}$$

and taking $k = n$, $a_{i,j} = K_{w,i-j}(c_j)$ in (5), we get

$$\sum_{i=n+1}^{\infty} \varphi_i \left\{ a \sum_{j=0}^{n} K_{w,i-j}(c_j) \right\} \leqslant \frac{1}{n+1} \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \varphi_i \{ a(n+1)^{1/s} K_{w,i-j}(c_j) \}. \tag{7}$$

Substituting (6) and (7) at the right-hand side of (4), we obtain after writing $b = 2^{1/s}a(n+1)^{1/s}$:

$$\begin{aligned}
&\rho(b2^{-1/s}(n+1)^{-1/s}(T_w x - x)) \\
&\quad \leqslant \tfrac{1}{2} \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} \varphi_i\{bK_{w,i-j}(c_j)\} \\
&\qquad + \tfrac{1}{2} \sum_{i=0}^{n} \varphi_i\{b\,|K_{w,0}(c_i) - c_i|\} \\
&\qquad + \frac{1}{n+1} \sum_{j=0}^{n} \sum_{i=n+1}^{\infty} \varphi_i\{bK_{w,i-j}(c_j)\} \\
&\quad \leqslant \tfrac{3}{2} \sum_{j=0}^{n} \sum_{i=1}^{\infty} \varphi_{i+j}\{bK_{w,i}(c_j)\} \\
&\qquad + \tfrac{1}{2} \sum_{i=0}^{n} \varphi_i\{b\,|K_{w,0}(c_i) - c_i|\},
\end{aligned}$$

thus getting the inequality (3).

The following conclusion is obtained from the lemma immediately:

THEOREM 2. *Let φ_i be s-convex, $0 < s \leqslant 1$, let $(1/c)\,K_{w,0}(c) \to^{\mathfrak{M}} 1$ for every $c > 0$ and let $\rho(bx_w^j) \to^{\mathfrak{M}} 0$ for $j = 0, 1, 2, \ldots,$ and all $b > 0$. Let S be the set of sequences of the form $x = c_0e_0 + c_1e_1 + \cdots + c_ne_n$. Then $T_w x \to^{\mathfrak{M}} x$ for every $x \in S$.*

Now, applying Theorems (T), 1, and 2 we get the following approximation theorem:

THEOREM 3. *Let φ_i be τ_+-bounded and s-convex, $0 < s \leqslant 1$, and let K be a singular kernel such that $\rho(bx_w^j) \to^{\mathfrak{M}} 0$ for $j = 0, 1, 2, \ldots,$ and for all $b > 0$. Let $T = (T_w)_{w \in \mathscr{W}}$ be the family of operators given by (1). Then $T_w x \to^{\rho, \mathfrak{M}} x$ for every $x \in l^{\varphi}$.*

In order to obtain a stronger thesis $T_w x \to^{\mathfrak{M}} x$, the following condition is of importance:

(δ_2) there exist positive numbers δ, K, and a sequence (a_i) of non-negative numbers such that $\sum_{i=0}^{\infty} a_i < \infty$ and the condition $\varphi_i(u) < \delta$ implies $\varphi_i(2u) \leqslant K\varphi_i(u) + a_i$ for $u \geqslant 0$, $i = 0, 1, 2, \ldots$.

It is known (see [3, 8.13 and 8.14]) that if (δ_2) holds, then ρ-convergence in l^{φ} is equivalent to norm convergence. The same holds if we replace usual convergence by convergence with respect to a filter. Therefore the following holds:

THEOREM 4. *Let the assumptions of Theorem 3 be satisfied and let* $\varphi = (\varphi_i)_{i=0}^{\infty}$ *satisfy the condition* (δ_2). *Then* $T_w x \to^{\mathfrak{M}} x$ *for every* $x \in l^{\varphi}$.

REFERENCES

1. J. MUSIELAK, Modular approximation by a filtered family of linear operators, *in* "Functional Analysis and Approximation, Proc. Conf. Oberwolfach, August 9–16, 1980," pp. 99–110, Birkhäuser, Basel, 1981.
2. J. MUSIELAK, On some approximation problems in modular spaces, *in* "Constructive Function Theory, Proc. Conf. Varna, June 1–5, 1981," pp. 455–461, Sofia, 1983.
3. J. MUSIELAK, Orlicz spaces and modular spaces, *in* "Lecture Notes in Mathematics Vol. 1034," pp. 1–222, Springer-Verlag, Berlin/Heidelberg/New York/Tokyo, 1983.